$$= \frac{(1-q)^2}{q-q^{m+1}} \cdot \frac{(1)}{1-q}$$

$$= \frac{1-q}{q(1-q^m)}$$

$$= \frac{1}{q(1+q+q^2+\ldots+q^{m-1})}.$$

ii)

$$\sum_{k=1}^{\infty} \frac{x^k}{[k][k+m]} = (1-q)^2 \sum_{k=1}^{\infty} \frac{x^k}{(1-q^k)(1-q^{k+m})}$$

$$= (1-q)^2 \sum_{k=1}^{\infty} x^k \left(\frac{1}{1-q^k} - \frac{1}{1-q^{k+m}}\right) \frac{1}{q^k - q^{k+m}}$$

$$= \frac{(1-q)^2}{1-q^m} \sum_{k=1}^{\infty} \left(\frac{x}{q}\right)^k \left(\frac{1}{1-q^k} - \frac{1}{1-q^{k+m}}\right)$$

$$= \frac{(1-q)^2}{1-q^m} \sum_{k=1}^{\infty} \left[x^k \left(\frac{1}{q^k} + \frac{1}{1-q^k}\right) - x^k q^m \left(\frac{1}{q^{k+m}} + \frac{1}{1-q^{k+m}}\right)\right]$$

$$= \frac{(1-q)^2}{1-q^m} \sum_{k=1}^{\infty} \left[x^k \left(\frac{1}{1-q^k}\right) - x^k q^m \left(\frac{1}{1-q^{k+m}}\right)\right]$$

$$= \frac{(1-q)}{1-q^m} \sum_{k=1}^{\infty} \frac{x^k}{[k]} - \frac{(1-q)^2 q^m}{(1-q^m)x^m} \sum_{k=1}^{\infty} x^{k+m} \left(\frac{1}{1-q^{k+m}}\right)$$

$$= \frac{(1-q)}{1-q^m} \sum_{k=1}^{\infty} \frac{x^k}{[k]} - \frac{(1-q)^2 q^m}{(1-q^m)x^m} \left[-\frac{x}{1-q} - \frac{x^2}{1-q^2} - \cdots - \frac{x^m}{1-q^m} + \sum_{k=1}^{\infty} x^k \left(\frac{1}{1-q^k}\right)\right]$$

$$= \frac{1-q}{1-q^m} \sum_{k=1}^{\infty} \frac{x^k}{[k]} + \frac{(1-q)^2 q^m}{(1-q^m)x^m} \left[\frac{x}{1-q} + \frac{x^2}{1-q^2} + \cdots + \frac{x^m}{1-q^m}\right] - \frac{(1-q)q^m}{(1-q^m)x^m} \sum_{k=1}^{\infty} \frac{x^k}{[k]}$$

$$= \frac{1-q}{1-q^m} \left(1 - \left(\frac{q}{x}\right)^m\right) \ln_q(x) + \frac{(1-q)q^m}{(1-q^m)} \left[\frac{x^{1-m}}{1} + \frac{x^{2-m}}{1+q} + \cdots + \frac{1}{1+q+q^2+\cdots q^{m-1}}\right].$$

Also solved by Arkady Alt, San Jose, CA, and the proposer.

- **5193:** *Proposed by Ovidiu Furdui, Cluj-Napoca, Romania*

Let f be a function which has a power series expansion at 0 with radius of convergence R.

a) Prove that $\displaystyle\sum_{n=1}^{\infty} n f^{(n)}(0) \left(e^x - 1 - \frac{x}{1!} - \frac{x^2}{2!} \cdots - \frac{x^n}{n!}\right) = \int_0^x e^{x-t} t f'(t) dt, \quad |x| < R.$

b) Let α be a non-zero real number. Calculate $\displaystyle\sum_{n=1}^{\infty} n\alpha^n \left(e^x - 1 - \frac{x}{1!} - \frac{x^2}{2!} \cdots - \frac{x^n}{n!}\right)$.

Solution 1 by Ángel Plaza, University of Las Palmas de Gran Canaria, Spain.

a) Let $S(x)$ be the sum of the series. Then, by differentiation, and for $|x| < R$,

$$S'(x) = \sum_{n=1}^{\infty} nf^{(n)}(0)\left(e^x - 1 - \frac{x}{1!} - \frac{x^2}{2!} \cdots - \frac{x^{n-1}}{(n-1)!}\right) = S(x) + \sum_{n=1}^{\infty} nf^{(n)}(0) \cdot \frac{x^n}{n!}.$$

It follows that $S'(x) = S(x) + xf'(x)$, and hence

$$S(x) = \int_0^x e^{x-t} tf'(t)dt + Ce^x,$$

where C is a constant of integration. Because $S(0) = 0$, we have $C = 0$ and

$$S(x) = \int_0^x e^{x-t} tf'(t)dt.$$

b) Note that if $f(x) = e^{\alpha x}$ then $f^{(n)}(0) = \alpha^n$, for $n \geq 1$. Hence, by part a), the sum of the given series is $\displaystyle\int_0^x e^{x-t} te^t dt = \frac{x^2 e^2}{2}$ if $\alpha = 1$. If $\alpha \neq 1$, the sum of the series is $\displaystyle\int_0^x e^{x-t} t\alpha e^{\alpha t} dt = \frac{\alpha x e^{\alpha x}}{\alpha - 1} + \frac{\alpha(e^x - e^{\alpha x})}{(\alpha - 1)^2}$.

Solution 2 by Anastasios Kotronis, Athens, Greece

a) From the problem's assumptions we have that

$$f(x) = \sum_{n=0}^{+\infty} \frac{f^{(n)}(0)}{n!} x^n \quad \text{and} \quad f'(x) = \sum_{n=1}^{+\infty} \frac{f^{(n)}(0)}{(n-1)!} x^{n-1} \quad \text{for} \quad |x| < R,$$

so, for $|x| < R$ we obtain

$$\begin{aligned} \int_0^x e^{x-t} tf'(t)\, dt &= \int_0^x e^{x-t} \sum_{n=1}^{+\infty} \frac{f^{(n)}(0)}{(n-1)!} t^n\, dt \\ &= e^x \sum_{n=1}^{+\infty} \frac{f^{(n)}(0)}{(n-1)!} \int_0^x t^n e^{-t}\, dt \\ &= e^x \sum_{n=1}^{+\infty} \frac{f^{(n)}(0)}{(n-1)!} I_n. \qquad (1) \end{aligned}$$

Now $I_n = -\displaystyle\int_0^x t^n (e^{-t})'\, dt = -x^n e^{-x} + nI_{n-1}$, so it is easily verified by induction that

$$I_n = -e^{-x}\left(x^n + nx^{n-1} + \cdots + n!x^0\right) + n!$$

With the above, (1) will give

$$
\begin{aligned}
\int_0^x e^{x-t} t f'(t)\, dt &= e^x \sum_{n=1}^{+\infty} \frac{f^{(n)}(0)}{(n-1)!} \left(-e^{-x}\left(x^n + nx^{n-1} + \cdots + n!x^0\right) + n!\right) \\
&= \sum_{n=1}^{+\infty} \frac{f^{(n)}(0)}{(n-1)!} \left(n!e^x - x^n - nx^{n-1} - \cdots - n!x^0\right) \\
&= \sum_{n=1}^{+\infty} n f^{(n)}(0) \left(e^x - 1 - \frac{x}{1!} - \cdots - \frac{x^n}{n!}\right).
\end{aligned}
$$

2) From (1) with $f(x) = e^{\alpha x}$ we obtained that

$$
\begin{aligned}
\sum_{n=1}^{+\infty} n\alpha^n \left(e^x - 1 - \frac{x}{1!} - \cdots - \frac{x^n}{n!}\right) &= \int_0^x e^{x-t} \alpha t e^{\alpha t}\, dt \\
&= I_\alpha.
\end{aligned}
$$

So,

$$
\begin{cases}
\displaystyle\int_0^x e^{x-t} t e^t \ dt = \frac{x^2 e^x}{2}, & \text{for } \alpha = 1 \\
I_\alpha = \alpha e^x \left(\int_0^x t\left(\frac{e^{(\alpha-1)t}}{\alpha-1}\right)\ dt\right), & \text{for} \alpha \neq 1 \\
\quad = \frac{\alpha e^{\alpha x}}{\alpha-1}\left(x - \frac{1}{\alpha-1}\right) + \frac{\alpha e^x}{(\alpha-1)^2}. &
\end{cases}
$$

Solution 3 by Arkady Alt, San Jose, CA

a) Let
$a_n(x) = e^x - 1 - \frac{x}{1!} - \frac{x^2}{2!} - ... - \frac{x^n}{n!}, n \in N \cup \{0\}$ and $F(x) = \sum_{n=1}^{\infty} n f^{(n)}(0)\, a_n(x).$
Noting that

$$
\begin{aligned}
a'_n(x) &= e^x - 1 - \frac{x}{1!} - \frac{x^2}{2!} - ... - \frac{x^{n-1}}{(n-1)!} \\
&= a_{n-1}(x), n \in N
\end{aligned}
$$

we obtain

$$
\begin{aligned}
F'(x) &= \left(\sum_{n=1}^{\infty} n f^{(n)}(0)\, a_n(x)\right)' \\
&= \sum_{n=1}^{\infty} n f^{(n)}(0)\, a'_n(x) \\
&= \sum_{n=1}^{\infty} n f^{(n)}(0)\, a_{n-1}(x).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\mathrm{F}(x) - F'(x) &= \sum_{n=1}^{\infty} n f^{(n)}(0)\left(a_n(x) - a_{n-1}(x)\right) \\
&= \sum_{n=1}^{\infty} n f^{(n)}(0)\left(-\frac{x^n}{n!}\right) \\
&= -\sum_{n=1}^{\infty} f^{(n)}(0)\frac{x^n}{(n-1)!} \\
&= -x\sum_{n=0}^{\infty} f^{(n+1)}(0)\frac{x^n}{n!} \\
&= -xf'(x).
\end{aligned}
$$

Multiplying equation $F'(x) - F(x) = xf'(x)$ by e^{-x} we obtain

$$
\begin{aligned}
F'(x)e^{-x} - F(x)e^{-x} &= e^{-x}xf'(x) \iff \left(F(x)e^{-x}\right)' \\
&= e^{-x}xf'(x).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
F(x)e^{-x} &= \int_o^x e^{-t}tf'(t)\,dt \\
&\iff F(x) = \int_o^x e^{x-t}tf'(t)\,dt.
\end{aligned}
$$

b) Let $f(x) = e^{\alpha x}$ then $f^{(n)}(0) = \alpha^n$ and, using the result we obtained in part (a) we get,

$$
\begin{aligned}
\sum_{n=1}^{\infty} n\alpha^n\left(e^x - 1 - \frac{x}{1!} - \frac{x^2}{2!} - \ldots - \frac{x^n}{n!}\right) &= \int_0^x e^{x-t}t\alpha e^{\alpha t}dt \\
&= \alpha e^x\int_0^x te^{t(\alpha-1)}dt.
\end{aligned}
$$

If $\alpha = 1$ then $\int_0^x te^{t(\alpha-1)}dt = \frac{x^2}{2}$ and, therefore,

$$
\begin{aligned}
\sum_{n=1}^{\infty} n\alpha^n\left(e^x - 1 - \frac{x}{1!} - \frac{x^2}{2!} - \ldots - \frac{x^n}{n!}\right) &= \sum_{n=1}^{\infty} n\left(e^x - 1 - \frac{x}{1!} - \frac{x^2}{2!} - \ldots - \frac{x^n}{n!}\right) \\
&= \frac{\alpha e^x x^2}{2}.
\end{aligned}
$$

If $\alpha \neq 1$ then

$$
\int_0^x te^{t(\alpha-1)}dt = \frac{xe^{(\alpha-1)x}}{\alpha - 1} - \frac{e^{(\alpha-1)x}}{(\alpha-1)^2}
$$

$$= \frac{e^{(\alpha-1)x}\left(x\left(\alpha-1\right)-1\right)}{\left(\alpha-1\right)^2}.$$

Hence,

$$\sum_{n=1}^{\infty} n\alpha^n \left(e^x - 1 - \frac{x}{1!} - \frac{x^2}{2!} - ... - \frac{x^n}{n!}\right) = \frac{\alpha e^{\alpha x}\left(x\left(\alpha-1\right)-1\right)}{\left(\alpha-1\right)^2}.$$

Solution 4 by Paolo Perfetti, Department of Mathematics, University "Tor Vergata," Rome, Italy

a) We need the two lemmas:

Lemma 1 $m!n! \le (n+m)!$

Proof by Induction. Let m be fixed. If $n=0$ evidently holds true. Let's suppose that the statement is true for any $1 \le n \le r$. For $n=r+1$ we have

$$m!(r+1)! = m!r!(r+1) \le (m+r)!(r+1) \le (m+r)!(m+r+1) = (m+r+1)!$$

which clearly holds for any $m \ge 0$. Since the inequality is symmetric, the induction on m proceeds along the same lines. q.e.d.

Lemma 2 The power series

$$\sum_{n=1}^{\infty} n f^{(n)}(0)\left(e^x - 1 - \frac{x}{1!} - \frac{x^2}{2!} - \ldots - \frac{x^n}{n!}\right) = \sum_{n=1}^{\infty} n f^{(n)}(0) \sum_{k=n+1}^{\infty} \frac{x^k}{k!}$$

converges for $|x| < R$ and is differentiable.

Proof:

$$\sum_{k=n+1}^{\infty} \frac{x^k}{k!} = \frac{x^{n+1}}{(n+1)!} \sum_{k=n+1}^{\infty} x^{k-n-1} \frac{(n+1)!}{k!}.$$

By using the Lemma 1 we can bound

$$\sum_{k=n+1}^{\infty} |x|^{k-n-1} \frac{(n+1)!}{k!} \le \sum_{k=n+1}^{\infty} \frac{|x|^{k-n-1}}{(k-n-1)!} = \sum_{k=0}^{\infty} \frac{|x|^k}{k!} = e^{|x|} \le e^R.$$

Thus we can write

$$\sum_{n=0}^{\infty} n|f^{(n)}(0)| \sum_{k=n+1}^{\infty} \frac{|x|^k}{k!} \le e^R |x| \sum_{n=0}^{\infty} n|f^{(n)}(0)| \frac{|x|^n}{n!} \frac{n!}{(n+1)!}$$

Since

$$\limsup_{n\to\infty} \left|\frac{f^{(n)}(0)}{n!}\right|^{1/n} = R^{-1} \implies \limsup_{n\to\infty} \left|\frac{f^{(n)}(0)}{n!}\frac{n}{n+1}\right|^{1/n} = R^{-1}$$

the series

$$\sum_{n=1}^{\infty} n f^{(n)}(0) \sum_{k=n+1}^{\infty} \frac{x^k}{k!}$$